Last time:

$$\dot{x} = f(x,u), \quad x(0) = x_0$$
$$y = h(x,u)$$

$$u \longrightarrow \boxed{\quad H \quad} \longrightarrow y$$

- H is $\mathcal{L}$-stable if

$$\|y_\tau\|_{\mathcal{L}} \leq \alpha(\|u_\tau\|_{\mathcal{L}}) + \beta \qquad \forall u \in \mathcal{L}_e^m, \ \forall \tau \geq 0$$

- H is $\mathcal{L}$-stable with finite gain if

$$\|y_\tau\|_{\mathcal{L}} \leq \gamma \|u_\tau\|_{\mathcal{L}} + \beta \qquad \forall u \in \mathcal{L}_e^m, \ \forall \tau \geq 0$$

- The smallest possible γ is called <u>gain</u>.

Computing L_2-gain:

- Linear sys.

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx + Du$$

- Assume A is Hurwitz $\Rightarrow$ exp stable $\Rightarrow$ L-stable
- We use freq. domain analysis to compute L_2-gain
- Laplace-transform: $\left(\hat{x}(s) = \int_0^\infty e^{-st} x(t)\,dt \right)$

$$s\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s)$$
$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

assume $x_0 = 0$

$$\Rightarrow \hat{y}(s) = G(s)\,\hat{u}(s) \quad \text{where}$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$\xrightarrow{s = j\omega} \underbrace{\hat{y}(j\omega)}_{\text{Fourier Transform (FT) of } y(t)} = G(j\omega)\,\hat{u}(j\omega)$$

$$\hat{y}(j\omega) = \int_0^\infty e^{-j\omega t} y(t)\,dt$$

Parseval's identity: for any signal $x \in L_2$

$$\int_0^{\infty} \|x(t)\|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\hat{x}(j\omega)\|^2 \, d\omega$$

$$\Longrightarrow \quad \|y\|_{L_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\hat{y}(j\omega)\|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \, G^*(j\omega) \, G(j\omega) \, \hat{u}(j\omega) \, d\omega$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_2^2 \, \|\hat{u}(j\omega)\|^2 \, d\omega$$

$$\leq \sup_{\omega} \|G(j\omega)\|_2^2 \, \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\hat{u}(j\omega)\|^2 \, d\omega$$

$$\leq \sup_{\omega} \|G(j\omega)\|_2^2 \, \|u\|_{L_2}^2$$

$$\Longrightarrow \quad \boxed{\|y\|_{L_2} \leq \underbrace{\left(\sup_{\omega} \|G(j\omega)\|_2 \right)}_{L_2\text{-gain}} \|u\|_{L_2}}$$

thm 5.4

also called: H_∞-norm in robust control

- It can be shown that this is the smallest possible number.

- Now consider an affine control system

$$\dot{x} = f(x) + \underbrace{G(x)}_{n \times m \text{ matrix}} u$$

$$y = h(x)$$

- Assume $f(0) = 0$ and $h(0) = 0$
- Assume there exists a Lyapunov function V and positive const. γ s.t.

$$\boxed{\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G(x)^T \frac{\partial V}{\partial x}^T + \frac{1}{2} \|h(x)\|^2 \leq 0 \quad \forall x} \quad (*)$$

- Then

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u$$

$$= \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \pm \frac{\gamma}{2} \|u\|^2 \pm \frac{1}{2\gamma} \left\| G(x)^T \frac{\partial V}{\partial x}^T \right\|^2$$

$$= \frac{\partial V}{\partial x} f(x) - \frac{1}{2} \left\| \sqrt{\gamma}\, u - \frac{1}{\sqrt{\gamma}} G(x)^T \frac{\partial V}{\partial x}^T \right\|^2 + \frac{\gamma}{2} \|u\|^2$$

$$+ \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G(x)^T \frac{\partial V}{\partial x}^T$$

$$\overset{(*)}{\leq} \frac{\gamma}{2} \|u\|^2 - \frac{1}{2} \|h(x)\|^2$$

Therefore, using $y = h(x)$

$$\dot{V}(x) \leq \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|y\|^2$$

- Integrating over a trajectory

$$V(x(t)) - V(x_0) \leq \frac{\gamma^2}{2}\int_0^t \|u(s)\|^2 dt - \frac{1}{2}\int_0^t \|y(s)\|^2 ds$$

$$\Longrightarrow \int_0^t \|y(s)\|^2 ds \leq \gamma^2 \int_0^t \|u(s)\|^2 ds + 2V(x_0)$$

$$\xrightarrow{t \to \infty} \|y\|_{L_2}^2 \leq \gamma^2 \|u\|_{L_2}^2 + 2V(x_0)$$

$$\xrightarrow[\text{and } \sqrt{a^2+b^2} \leq a+b]{\text{taking } \sqrt{\ }} \|y\|_{L_2} \leq \gamma \|u\|_{L_2} + \sqrt{2V(x_0)}$$

$\Longrightarrow$ L_2-stable with gain $\leq \gamma$

Thm. 5.5

- The ineq. for V is called Hamilton-Jacobi ineq.

- It is possible to find V that satisfies HJ ineq. even if the sys. $\dot{x} = f(x)$ is not exp. stable.

Example :

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_1^3 - k x_2 + u$$
$$y = x_2$$

- It is affine in control ✓

- $V(x) = \frac{1}{2} x_2^2 + \frac{1}{4} a x_1^4$ is a Lyapunov function for unforced system (it is the energy)

- we try $V(x) = b\left(\frac{1}{2} x_2^2 + \frac{a x_1^4}{4}\right)$ for $b \geq 0$ as candidate that solves HJ ineq.

• $$\frac{\partial V}{\partial x} f(x) = b\,[a x_1^3,\ x_2] \begin{bmatrix} x_2 \\ -a x_1^3 - k x_2 \end{bmatrix}$$
$$= -b k x_2^2$$

• $$\frac{\partial V}{\partial x} G(x) = b\,[a x_1^3,\ x_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= b x_2$$

$$\Rightarrow \quad \frac{\partial V}{\partial x} G_{(x)} G_{(x)}^T \frac{\partial V}{\partial x}^T = (b x_2)^2 = b^2 x_2^2$$

- $\|h(x)\|^2 = x_2^2$

HJ ineq.

$$\Rightarrow \quad -bk x_2^2 + \frac{1}{2\gamma^2} b^2 x_2^2 + \frac{1}{2} x_2^2 \leq 0$$

$$\Rightarrow \quad \left(-bk + \frac{b^2}{2\gamma^2} + \frac{1}{2}\right) x_2^2 \leq 0$$

- So, we need to choose $\gamma, b > 0$ s.t.

$$-bk + \frac{b^2}{2\gamma^2} + \frac{1}{2} \leq 0$$

- we choose b that minimizes the LHS

$$-k + \frac{b}{\gamma^2} = 0 \Rightarrow b = k\gamma^2$$

- Therefore, with $b = k\gamma^2$

$$-\frac{k^2\gamma^2}{2} + \frac{1}{2} \leq 0 \Rightarrow \gamma^2 \geq \frac{1}{k^2}$$

- we choose $\gamma = \frac{1}{k} \Rightarrow L_2\text{-gain} \leq \frac{1}{k}$

KYP or bounded real lemma:

Kalman - Popov - Yakubovich

- Consider the linear sys.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- We saw that $\sup_{\omega} \|H(j\omega)\|_2$ is the L_2-gain

- HJ inequality provides another way to compute the L_2-gain.

- Let $V(x) = \frac{1}{2} x^T P x$. Then HJ ineq is

$$\frac{1}{2} x^T (PA + A^T P) x + \frac{1}{2\gamma^2} x^T P B B^T P x + \frac{1}{2} x^T C^T C x \leq 0 \quad \forall x$$

$$\Longleftrightarrow \quad PA + A^T P + \frac{1}{\gamma^2} PB B^T P + C^T C \preceq 0$$

negative-def matrix

- So if we can find a p.d. matrix P that solves the ineq for γ, the system has L_2 gain smaller than γ.

- The problem of finding the smallest constant $\gamma > 0$ can be formulated as an optimization problem

$$\min_{\substack{\gamma > 0 \\ P \succ 0}} \gamma \quad \text{s.t.}$$

$$PA + A^TP + PB^TBP + C^TC < 0$$

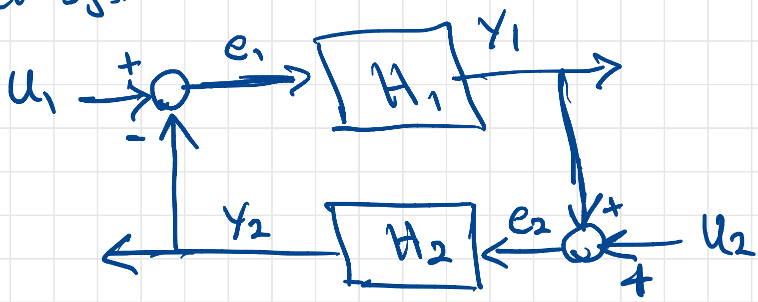
negative definite

- This is a convex problem actually a semi-definite programming (SDP)

- It is another way to compute the L_2 gain of a linear system.

Feedback sys: Small-gain thm

- Feedback sys.



- Assume the feedback sys is well-defined:

for every input $u_1, u_2 \in L_e$, there exists well-defined output $y_1, y_2 \in L_e$

- Overall system

$$\text{input: } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{output: } y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Assume H_1 and H_2 are L-stable with finite-gain

$$\|y_1\|_L \le \gamma_1 \|e_1\|_L + \beta_1$$

$$\|y_2\|_L \le \gamma_2 \|e_2\|_L + \beta_2$$

- Question: is the overall sys L-stable?

Small-gain thm:

- Feedback connection is finite-gain L-stable if $\gamma_1 \gamma_2 < 1$.

proof:

$$e_1 = u_1 - y_2$$
$$e_2 = u_2 + y_1$$

$$\Rightarrow \quad \|e_1\|_L \le \|u_1\|_L + \|y_2\|_L$$
$$\le \|u_1\|_L + \gamma_2 \|e_2\|_L + \beta_2$$

Similarly

$$\|e_2\|_L \le \|u_2\|_L + \|y_1\|_L$$
$$\le \|u_2\|_L + \gamma_1 \|e_1\|_L + \beta_1$$

$$\Rightarrow \quad \|e_1\|_L \le \|u_1\|_L + \gamma_2 \|u_2\|_L + \gamma_1\gamma_2 \|e_1\|_L + \beta_2 + \gamma_2 \beta_1$$

$$\overset{\gamma_1\gamma_2<1}{\Longrightarrow} \quad \|e_1\|_L \le \frac{1}{1-\gamma_1\gamma_2}\left(\underbrace{\|u_1\|_L + \gamma_2 \|u_2\|_L}_{\le (1+\gamma_2)\|u\|_L} + \beta_2 + \gamma_2 \beta_1\right)$$

Similarly

$$\|e_2\| \leq \frac{1}{1-\gamma_1\gamma_2}\left(\underbrace{\|u_2\| + \gamma_1\|u_1\|}_{\leq (1+\gamma_1)\|u\|_L} + \beta_1 + \gamma_1\beta_2\right)$$

$$\Longrightarrow \quad \|e\|_L \leq \|e_1\|_L + \|e_2\|_L$$

$$\leq \frac{2+\gamma_1+\gamma_2}{1-\gamma_1\gamma_2}\|u\|_L + \beta$$

$$\beta = \frac{(1+\gamma_2)\beta_1 + (1+\gamma_1)\beta_2}{1-\gamma_1\gamma_2}$$

$$\Longrightarrow \quad \|y\|_L \leq \|e\|_L + \|u\|_L$$

$$\leq \left(\frac{2+\gamma_1+\gamma_2}{1-\gamma_1\gamma_2}+1\right)\|u\|_L + \beta$$

$\Longrightarrow$ L-stable with finite gain.